Steve Silverman's Question

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The question posed: In yesterday's seminar ("Introduction to the Quantum Theory of Open Systems") I made passing reference to Schrödinger's identity

$$(\Delta A)^2(\Delta B)^2 \geqslant \Big\langle \frac{\mathsf{A\,B} - \mathsf{B\,A}}{2i} \Big\rangle^2 + \Big[\Big\langle \frac{\mathsf{A\,B} + \mathsf{B\,A}}{2} \Big\rangle - \langle \mathsf{A} \rangle\langle \mathsf{B} \rangle\Big]^2 \tag{1}$$

where $\langle \mathsf{A} \rangle \equiv (\psi|\mathsf{A}|\psi)$ and $\Delta A \equiv \sqrt{\langle \mathsf{A}^2 \rangle - \langle \mathsf{A} \rangle^2}$. In the question period, Steve Silverman drew attention to the fact that if $|\psi)$ is an eigenvector of A

$$\mathsf{A}|\psi) = \lambda|\psi) \tag{2}$$

then $\Delta A = 0$, and asked: How can (1) remain valid under circumstances which cause the expression $(\Delta A)^2(\Delta B)^2$ on the left to vanish?

. . . **and resolved:** It must be that circumstances that serve to kill the expression on the left side of (1) serve also to kill the expression on the right, which is easily seen to be the case: drawing upon (2) we have

$$\text{right side of (1)} = \lambda^2 \Big\langle \frac{\mathsf{B} - \mathsf{B}}{2i} \Big\rangle^2 + \Big[\lambda \Big\langle \frac{\mathsf{B} + \mathsf{B}}{2} \Big\rangle - \lambda\langle \mathsf{B} \rangle\Big]^2 = 0$$

Sharpened version of the same question: Send $\mathsf{A} \rightarrow \mathsf{x}$ and $\mathsf{B} \rightarrow \mathsf{p}$, whereupon (1) becomes

$$\begin{aligned}(\Delta x)^2(\Delta p)^2 &\geqslant \Big\langle \frac{\mathsf{x\,p} - \mathsf{p\,x}}{2i} \Big\rangle^2 + \Big[\Big\langle \frac{\mathsf{x\,p} + \mathsf{p\,x}}{2} \Big\rangle - \langle \mathsf{x} \rangle\langle \mathsf{p} \rangle\Big]^2 \\ &= (\hbar/2)^2 + \Big[\Big\langle \frac{\mathsf{x\,p} + \mathsf{p\,x}}{2} \Big\rangle - \langle \mathsf{x} \rangle\langle \mathsf{p} \rangle\Big]^2 \end{aligned} \tag{3}$$

The leading term on the right never vanishes, so we would appear to have a problem if it made sense to write $\mathsf{x}|\psi) = \lambda|\psi)$. But it doesn't: the x-spectrum is continuous, its eigenfunctions are distributions and $\mathsf{x}|\psi) = \lambda|\psi)$ makes sense only in the shade of an integral sign. We might temper the situation by introducinig the square root of a Gaussian

$$\psi_\sigma(x-a) = \frac{1}{\sqrt{\sqrt{2\pi}\sigma}}\exp\left\{-\left(\frac{x-a}{2\sigma}\right)^2\right\}$$

the square of which $\longrightarrow \delta(x-a)$ as $\sigma \downarrow 0$. Setting $\hbar = 1$, we then compute

$$\begin{aligned}\langle \mathsf{x}\rangle &= a\\ \langle \mathsf{x}^2\rangle &= a^2+\sigma^2\\ \langle \mathsf{p}\rangle &= 0\\ \langle \mathsf{p}^2\rangle &= \tfrac{1}{4}\sigma^{-2}\\ \left\langle\frac{\mathsf{x}\,\mathsf{p}-\mathsf{p}\,\mathsf{x}}{2i}\right\rangle &= \tfrac{1}{2}\\ \left\langle\frac{\mathsf{x}\,\mathsf{p}+\mathsf{p}\,\mathsf{x}}{2}\right\rangle &= 0\end{aligned}$$

We therefore have

$$\left.\begin{aligned}(\Delta x)^2(\Delta p)^2 = (a^2+\sigma^2-a^2)(\tfrac{1}{4}\sigma^{-2}) &= \tfrac{1}{4}\\ \text{right side of (3)} = (\tfrac{1}{2})^2 + [0]^2 &= \tfrac{1}{4}\end{aligned}\right\}\quad:\quad\text{all }\sigma$$